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Zero-measure Cantor spectrum for Schrödinger operators with low-complexity potentials

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Abstract

We consider discrete one-dimensional Schrödinger operators whose potentials belong to minimal subshifts of low combinatorial complexity and prove for a large class of such operators that the spectrum is a Cantor set of zero Lebesgue measure. This is obtained through an analysis of the frequencies of the subwords occurring in the potential. Our results cover most circle map and Arnoux–Rauzy potentials.

Résumé

Nous considérons des opérateurs unidimensionnels discrets de Schrödinger dont les potentiels appartiennent aux « subshifts » minimaux de basse complexité combinatoire. Nous démontrons pour une grande classe de tels opérateurs que le spectre est un ensemble de Cantor de mesure nulle de Lebesgue. Ceci est obtenu par une analyse des fréquences des facteurs intervenant dans le potentiel. Nos résultats couvrent la plupart des potentiels résultant des codages des rotations et des « subshifts » d'Arnoux–Rauzy.

MSC: 37A30; 47B39

Keywords: Schrödinger operators; Cantor spectrum; Multiplicative ergodic theorem; Uniform cocycles

1. Introduction

This paper is concerned with Schrödinger operators,

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n), \tag{1}$$

acting in the Hilbert space $\ell^2(\mathbb{Z})$. The case of a periodic potential, $V(n) = V(n+p)$ for some p and all n, is classical. It follows from Floquet theory that the spectrum consists of p closed intervals whose interiors are mutually disjoint.

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Thus, the spectrum has $p-1$ gaps, some of which may be empty or “closed”. Generically, however, these gaps are open.

A potential is almost periodic if it is close to being periodic in a suitable sense. Note that the family $\Omega_0(V) = \{T^m V\colon m \in \mathbb{Z}\}$, where $[TV](n) = V(n+1)$, is finite if and only if V is periodic. The standard notion of almost periodicity corresponds to the requirement that the closure of $\Omega_0(V)$ in $\ell^\infty(\mathbb{Z})$ is compact.

We will be interested in models typically studied in the context of quasicrystals, where the potentials take finitely many values and hence a reasonable notion of almost periodicity cannot be defined in the way above. Rather, almost periodicity in this context is defined as follows (cf. [71]). If V takes values in the finite set $A \subset \mathbb{R}$, then V is said to be almost periodic if every subword of V occurs with bounded gaps. Here, the subwords of V are the finite words $V(n)\cdots V(n+k-1)$, where $n \in \mathbb{Z}$ denotes the position and $k \geqslant 1$ denotes the length, and a subword w of V of length k occurs with bounded gaps if the set $\{n \in \mathbb{Z}\colon V(n)\cdots V(n+k-1) = w\}$ is relatively dense in $\mathbb{Z}$. Equivalently, the closure $\Omega(V)$ of $\Omega_0(V)$ in $A^{\mathbb{Z}}$, equipped with product topology, together with the shift T forms a minimal topological dynamical system, called the subshift generated by V.

Since the spectra of periodic operators generically have many gaps, it is widely expected that the spectra of almost periodic operators typically have infinitely many gaps or, more strongly, a dense set of gaps. That is, the spectra should have a tendency to be Cantor sets.

Proving Cantor spectrum for almost periodic operators has been the objective of many papers during the last two decades. For potentials that are ℓ^∞-almost periodic, the issue of Cantor spectrum was discussed, for example, in [5,6,10,17,35,45,57,69], while for potentials taking finitely many values, results in this direction may be found in [1,7–9,15,23,24,30,56,59,60,67,77].

In fact, in the latter case, all results actually establish that the spectrum has zero Lebesgue measure, from which the Cantor property follows immediately. The key to understanding these results is a remarkably general theorem of Kotani which states that for aperiodic ergodic potentials taking finitely many values, the Lyapunov exponent is positive for almost every energy [55]. Note that $\Omega(V)$ always admits an ergodic measure, and in many cases of interest, this measure is unique. Moreover, due to minimality, the spectrum of the operator is the same for all potentials in $\Omega(V)$. Thus, zero-measure spectrum follows once one shows that the Lyapunov exponent vanishes for energies in the spectrum.

There are two approaches to proving the statement just mentioned. The first approach considers energies in the spectrum and wants to establish a vanishing Lyapunov exponent. This approach was first employed by Sütő [77] and Bellissard et al. [9], and then used in many subsequent papers. An important tool in this context is the trace map, which is a dynamical system on $\mathbb{R}^d$ that describes the evolution of transfer matrix traces along the various levels of the hierarchy in potentials having a so-called S-adic structure. This is also the reason why this approach is limited to such potentials and hence is applicable only in certain situations. A second approach was developed by Lenz [59] and it is based on uniform convergence in the sub-additive ergodic theorem. The spectrum consists in general of those energies for which either the Lyapunov exponent is zero or it is positive with non-uniform convergence. Thus, a proof of uniformity for all energies shows that the exponent must indeed vanish for all energies in the spectrum. Lenz also found a general characterization of uniform convergence in general sub-additive situations [58] which allowed him to prove zero-measure Cantor spectrum for all potentials generated by primitive substitutions and, more generally, all potentials satisfying a positive-weights condition. Subsequently, we considered the special case of $\mathrm{SL}(2,\mathbb{R})$ cocycles over $\Omega(V)$, which is the relevant case for the study of Schrödinger operators, and found that a condition of Boshernitzan [14] is sufficient for the desired uniform convergence [25].

In this paper we will establish zero-measure Cantor spectrum for large classes of potentials taking finitely many values by proving Boshernitzan's condition for the associated subshifts. We will also verify Boshernitzan's condition for all potentials for which the trace map approach had been successfully employed, and hence find a new proof of zero-measure Cantor spectrum for these models. Consequently, the approach based on uniform convergence to the Lyapunov exponent has a larger scope than the one based on trace maps. It should, however, be noted that the trace map approach has additional merits in that its output can often be used to obtain additional information about the spectral type [7–9,15,22,23,56,67,76,77], fine properties of the spectral measures [19,21,44], and quantum dynamical quantities such as transport exponents [20,26–29,53].

Let us now describe our results. One class of potentials we will be especially interested in are those arising from so-called circle maps. They have the form:

$$V(n) = \sum_{m=1}^{p} v_m \chi_{I_m}(n\alpha + \theta \mod 1),$$

where α is irrational, $\theta \in [0, 1)$, and $I_1, \ldots, I_M$ form a partition of the torus $[0, 1)$ into half-open intervals, that is, $I_m = [\beta_{m-1}, \beta_m)$ and $0 = \beta_0 < \beta_1 < \cdots < \beta_p = 1$. The special case,

$$V(n) = \lambda \chi_{[0,\alpha)}(n\alpha + \theta \mod 1),$$

yields the class of Sturmian potentials. The Sturmian potentials with $\alpha = (\sqrt{5} - 1)/2$ are called Fibonacci potentials. They have an alternate description in terms of an iterated substitution rule. Zero-measure spectrum has been established for all Sturmian potentials; see Sütő [77] for Fibonacci and Bellissard et al. [9] for the general result. These results were obtained by means of the trace map approach, which seems to be limited to this case. In this paper we prove zero-measure spectrum for two large classes of circle map potentials: in the two-interval case, $p = 2$, for almost every β_1 with respect to the Lebesgue measure on $(0, 1)$, and in the general case, $p \geqslant 2$, it suffices that each of the points $\{\beta_1, \ldots, \beta_{p-1}\}$ be rational. These results are independent of the values the potential takes on the intervals of the partition as long as V is non-constant, that is, the set $\{v_1, \ldots, v_p\}$ has cardinality at least two.

Sturmian sequences can also be characterized by their subword structure. While this will be explained in more detail below, we mention at this point that the characteristic property is the existence, for every given length k, of a unique word that has multiple extensions to the right, and the number of distinct extensions for these words is exactly two for every length. If one replaces two by a larger integer, one essentially obtains the Arnoux–Rauzy sequences, whose initial study was performed by Arnoux and Rauzy in [4]. We will show that almost all Arnoux–Rauzy potentials (with respect to a natural measure on sequences of this type) lead to zero measure spectrum.

There is another point of view that motivates our study. As explained above, one should expect many gaps in the spectrum if the potential is close to being periodic. Periodic sequences may be characterized in terms of their combinatorial complexity. Let $p(n)$ denote the number of mutually distinct subwords of V that have length n. The function $p : \mathbb{Z}_+ \to \mathbb{Z}_+$ is called the combinatorial complexity function associated with V. It is easy to see that p is bounded if V is periodic. On the other hand, if there is some n_0 such that $p(n_0) \leqslant n_0$, then V must be periodic. That is, aperiodic sequences satisfy the universal $p(n) \geqslant n + 1$ for every n. Thus, from a combinatorial perspective, a potential V is close to being periodic if its complexity function $p(n)$ does not grow much faster than $n + 1$, for example, if satisfies a linear upper bound. A natural goal therefore could be to prove zero-measure Cantor spectrum for such potentials. What we in essence achieve in this paper is to prove such a result for all or almost all members of the standard classes of almost periodic potentials whose complexity function is linearly bounded above. As a consequence of our results, it is seen that Cantor spectra are prevalent among these potentials. The Boshernitzan condition is therefore a very useful tool in this context, as the trace map approach is limited to a much smaller class of potentials.

Our approach also suggests a number of open problems, centered around the following question: When do spectra cease to be of Cantor type? One could either increase combinatorial complexity, or replace the specific classes treated in this paper by other specific examples; for example, torus maps or models associated with a skew-shift on the torus. The study of these examples would also shed some light on their combinatorial and dynamical properties, issues that have not been studied in the same depth as in the case of their circle analog.

Given the results of [59] and [25], we may focus our attention on proving Boshernitzan's condition for a given aperiodic subshift. It then follows that for each element of the subshift, the corresponding Schrödinger operator has zero-measure Cantor spectrum. We will describe Boshernitzan's condition in Section 2 along with a summary of the main results of [25]. In the subsequent sections we discuss the standard classes of minimal subshifts of low combinatorial complexity and explore the validity of Boshernitzan's condition for each of them.

2. Boshernitzan's condition and zero-measure spectrum

(Ω, T) is called a subshift over $\mathcal{A}$ if $\mathcal{A} \subset \mathbb{R}$ is finite with discrete topology and Ω is a closed T-invariant subset of $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}^{\mathbb{Z}}$ carries the product topology and $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is given by $[Ts](n) = s(n + 1)$. (Ω, T) gives rise to the family $(H_\omega)_{\omega \in \Omega}$, where H_ω is an operator of the form (1) with potential given by the real-valued sequence ω.

We will freely use notions from combinatorics on words (see, e.g., [61,62]). In particular, the elements of $\mathcal{A}$ are called letters and the elements of the free monoid $\mathcal{A}^*$ over $\mathcal{A}$ are called words. The length $|w|$ of a word w is the number of its letters. The number of occurrences of a word w in a word x is denoted by $\#_w(x)$.

Each subshift (Ω, T) over $\mathcal{A}$ gives rise to the associated set of words:

$$\mathcal{W}(\Omega) := \big\{\omega(k)\cdots\omega(k+n-1)\colon\ k \in \mathbb{Z},\ n \in \mathbb{N},\ \omega \in \Omega\big\}. \tag{2}$$

For $w \in \mathcal{W}$, we define:

$$V_w := \big\{\omega \in \Omega\colon\ \omega(1)\cdots\omega\big(|w|\big) = w\big\}.$$

Finally, if ν is a T-invariant probability measure on (Ω, T) and $n \in \mathbb{N}$, we set:

$$\eta_\nu(n) := \min\big\{\nu(V_w)\colon\ w \in \mathcal{W},\ |w| = n\big\}. \tag{3}$$

If (Ω, T) is uniquely ergodic with invariant probability measure μ, we set $\eta(n) := \eta_\mu(n)$.

Definition 2.1. Let (Ω, T) be a subshift over $\mathcal{A}$. Then, (Ω, T) is said to satisfy condition (B) if there exists an ergodic probability measure ν on Ω with

$$\limsup_{n\to\infty} n\eta_\nu(n) > 0.$$

Thus, (Ω, T) satisfies (B) if and only if there exists an ergodic probability measure ν on Ω, a constant $C > 0$ and a sequence (l_n) in $\mathbb{N}$ with $l_n \to \infty$ for $n \to \infty$ such that $|w|\nu(V_w) \geqslant C$ whenever $w \in \mathcal{W}(\Omega)$ with $|w| = l_n$ for some $n \in \mathbb{N}$.

This condition was introduced by Boshernitzan in [14] as a sufficient condition for unique ergodicity of the subshift (Ω, T) (see [25] for an alternate proof). It is related to the problem of proving Cantor spectrum by the following theorem; see [25, Theorem 2].

Theorem 1. *Let (Ω, T) be a minimal subshift which satisfies* (B). *If (Ω, T) is aperiodic, then there exists a Cantor set $\Sigma \subset \mathbb{R}$ of zero Lebesgue measure with $\sigma(H_\omega) = \Sigma$ for every $\omega \in \Omega$.*

When studying the validity of (B) for a given subshift, it is very useful to have stability of this property with respect to morphic images and pre-images of the subshift. Properties of this kind were also established in [25]. Since we will need these results below, let us recall them. Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. A map $S : \mathcal{A} \to \mathcal{B}^*$ is called a substitution. S can be extended to $\mathcal{A}^*$ in the obvious way. Moreover, for a two-sided infinite word $(\omega(n))_{n\in\mathbb{Z}}$ over $\mathcal{A}$, we can define $S(\omega)$ by:

$$S(\omega) = \cdots S\big(\omega(-2)\big)S\big(\omega(-1)\big)|S\big(\omega(0)\big)S\big(\omega(1)\big)S\big(\omega(2)\big)\cdots,$$

where | denotes the position of zero. If (Ω, T) is a subshift over A and $S : \mathcal{A} \to \mathcal{B}^*$ is a substitution, we define $\Omega(S)$ by:

$$\Omega(S) = \big\{T^k S(\omega)\colon\ \omega \in \Omega,\ k \in \mathbb{Z}\big\}.$$

Then, $(\Omega(S), T)$ is a subshift over $\mathcal{B}$. It is not hard to see that $(\Omega(S), T)$ is minimal (uniquely ergodic) if Ω is minimal (uniquely ergodic).

The substitution S is called *recognizable* (with respect to (Ω, T)) if there exists a locally constant map,

$$\widetilde{S} : \Omega(S) \to \Omega \times \mathbb{Z},$$

with $\widetilde{S}(T^k S(\omega)) = (\omega, k)$, whenever $0 \leqslant k \leqslant |S(\omega(0))|$. Recognizability is known for various classes of substitutions that generate aperiodic subshifts, including all primitive substitutions [66] and all substitutions of constant length that are one-to-one [3] (cf. the discussion in [39]).

Theorem 8 of [25] shows the following:

Theorem 2. *Let (Ω, T) be a minimal uniquely ergodic subshift over $\mathcal{A}$. Let S be a substitution on $\mathcal{A}$ and $(\Omega(S), T)$ the corresponding subshift.*

(a) *If* (Ω, T) *satisfies* (B)*, so does* $(\Omega(S), T)$.
(b) *If* $(\Omega(S), T)$ *satisfies* (B) *and* S *is recognizable, then* (Ω, T) *satisfies* (B) *as well.*

3. Sturmian and quasi-Sturmian subshifts

Consider a minimal subshift (Ω, T) over $\mathcal{A}$. Recall that the associated set of words is given by:

$$\mathcal{W}(\Omega) := \{\omega(k)\cdots\omega(k+n-1):\ k \in \mathbb{Z},\ n \in \mathbb{N},\ \omega \in \Omega\}.$$

The (factor) complexity function $p : \mathbb{N} \to \mathbb{N}$ is then defined by:

$$p(n) = \#\mathcal{W}_n(\Omega), \tag{4}$$

where $\mathcal{W}_n(\Omega) = \mathcal{W}(\Omega) \cap \mathcal{A}^n$ and # denotes cardinality.

It is a fundamental result of Hedlund and Morse that periodicity can be characterized in terms of the complexity function [42]:

Theorem 3 *(*Hedlund–Morse*).* (Ω, T) *is aperiodic if and only if* $p(n) \geqslant n+1$ *for every* $n \in \mathbb{N}$.

Aperiodic subshifts of minimal complexity, $p(n) = n + 1$ for every $n \in \mathbb{N}$, exist and they are called Sturmian. If the complexity function satisfies $p(n) = n + k$ for $n \geqslant n_0$, $k, n_0 \in \mathbb{N}$, the subshift is called quasi-Sturmian. It is known that quasi-Sturmian subshifts are exactly those subshifts that are a morphic image of a Sturmian subshift; compare [16,18,68].

There are a large number of equivalent characterizations of Sturmian subshifts. We are mainly interested in their geometric description in terms of an irrational rotation. Let $\alpha \in (0,1)$ be irrational and consider the rotation by α on the circle,

$$R_\alpha : [0,1) \to [0,1), \qquad R_\alpha\theta = \{\theta + \alpha\},$$

where $\{x\}$ denotes the fractional part of x, $\{x\} = x \mod 1$. The coding of the rotation R_α according to a partition of the circle into two half-open intervals of length α and $1-\alpha$, respectively, is given by the sequences:

$$v_n(\alpha, \theta) = \chi_{[0,\alpha)}\big(R_\alpha^n \theta\big).$$

We obtain a subshift,

$$\Omega_\alpha = \overline{\{v(\alpha,\theta):\ \theta \in [0,1)\}} = \{v(\alpha,\theta):\ \theta \in [0,1)\} \cup \{\tilde{v}^{(k)}(\alpha):\ k \in \mathbb{Z}\} \subset \{0,1\}^{\mathbb{Z}},$$

which can be shown to be Sturmian. Here, $\tilde{v}_n^{(k)}(\alpha) = \chi_{(0,\alpha]}(R_\alpha^{n+k} 0)$. Conversely, every Sturmian subshift is essentially of this form, that is, if Ω is minimal and has complexity function $p(n) = n + 1$, then up to a one-to-one morphism, $\Omega = \Omega_\alpha$ for some irrational $\alpha \in (0,1)$.

By uniform distribution, the frequencies of factors of Ω are given by the Lebesgue measure of certain subsets of the torus. Explicitly, if we write $I_0 = [0,\alpha)$ and $I_1 = [\alpha, 1)$, then the word $w = w_1 \dots w_n \in \{0,1\}^n$ occurs in $v(\alpha,\theta)$ at site $k+1$ if and only if

$$\{k\alpha + \theta\} \in I(w_1, \dots, w_n) := \bigcap_{j=1}^{n} R_\alpha^{-j}(I_{w_j}).$$

This shows that the frequency of w is θ-independent and equal to the Lebesgue measure of $I(w_1, \dots, w_n)$. It is not hard to see that $I(w_1, \dots, w_n)$ is an interval whose boundary points are elements of the set:

$$P_n(\alpha) := \{\{-j\alpha\}:\ 0 \leqslant j \leqslant n\}.$$

The $n+1$ points of $P_n(\alpha)$ partition the torus into $n+1$ subintervals and hence the length $h_n(\alpha)$ of the smallest of these intervals bounds the frequency of a factor of length n from below. It is therefore of interest to study $\limsup n h_n(\alpha)$.

To this end we recall the notion of a continued fraction expansion; compare [52,74]. For every irrational $\alpha \in (0,1)$, there are uniquely determined $a_k \in \mathbb{N}$ such that

$$\alpha = [a_1, a_2, a_3, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}. \tag{5}$$

The associated rational approximants $\frac{p_k}{q_k}$ are defined by:

$$p_0 = 0, \quad p_1 = 1, \quad p_k = a_k p_{k-1} + p_{k-2},$$
$$q_0 = 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2}.$$

These rational numbers are best approximants to α in the following sense,

$$\min_{\substack{p,q \in \mathbb{N} \\ 0<q<q_{k+1}}} |q\alpha - p| = |q_k \alpha - p_k|, \tag{6}$$

and the quality of the approximation can be estimated according to

$$\frac{1}{q_k + q_{k+1}} < |q_k \alpha - p_k| < \frac{1}{q_{k+1}}. \tag{7}$$

By definition, we have:

$$h_n(\alpha) = \min_{0<|q|\leqslant n} \{q\alpha\}.$$

Notice that for $0 < q \leqslant n$, we have $\min\{\{q\alpha\}, \{-q\alpha\}\} = \|q\alpha\|$, where we denote $\|x\| = \min_{p\in\mathbb{Z}} |x - p|$.

In particular,

$$h_n(\alpha) = \min_{0<q\leqslant n} \|q\alpha\|. \tag{8}$$

As noted by Hartman [41], this shows that $h_n(\alpha)$ can be expressed in terms of the continued fraction approximants. Indeed, if we combine (6) and (8), we obtain:

Lemma 3.1 *(*Hartman*). If $q_k \leqslant n < q_{k+1}$, then*

$$h_n(\alpha) = |q_k \alpha - p_k|.$$

This allows us to show the following:

Theorem 4. *Every Sturmian subshift obeys the Boshernitzan condition* (B).

Proof. We only need to show that

$$\limsup_{n\to\infty} n h_n(\alpha) \geqslant C > 0. \tag{9}$$

We shall verify this on the subsequence $n_k = q_{k+1} - 1$. Hartman's lemma together with (7) shows that

$$n_k h_{n_k}(\alpha) = (q_{k+1} - 1)|q_k \alpha - p_k| \geqslant \frac{q_{k+1} - 1}{q_k + q_{k+1}} = \frac{1 - q_{k+1}^{-1}}{1 + q_k q_{k+1}^{-1}}.$$

Thus (9) holds (with $C = 1/3$, say). $\square$

Corollary 1. *Every quasi-Sturmian subshift obeys* (B).

Proof. This follows from Theorem 4 along with the stability result, Theorem 2. $\square$

Remark. Theorem 4 may also be obtained as a consequence of a result due to Boshernitzan; see Theorem 6 below. However, we decided to give the short proof above because we introduced quantities and ideas along the way that will prove useful in our study of the circle map case in Section 5.

4. Interval exchange transformations

Subshifts generated by interval exchange transformations (IETs) are natural generalizations of Sturmian subshifts. They were studied, for example, in [13,36–38,49–51,63,72,78–80].

IETs are defined as follows. Given a probability vector $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_i > 0$ for $1 \leqslant i \leqslant m$, we let $\mu_0 = 0$, $\mu_i = \sum_{j=1}^{i} \lambda_j$, and $I_i = [\mu_{i-1}, \mu_i)$. Let τ be a permutation of $\mathcal{A}_m = \{1, \dots, m\}$, that is, $\tau \in S_m$, the symmetric group. Then $\lambda^\tau = (\lambda_{\tau^{-1}(1)}, \dots, \lambda_{\tau^{-1}(m)})$ is also a probability vector and we can form the corresponding μ_i^τ and I_i^τ. Denote the unit interval $[0, 1)$ by I. The (λ, τ) interval exchange transformation is then defined by:

$$T : I \to I, \quad T(x) = x - \mu_{i-1} + \mu^\tau_{\tau(i)-1} \text{ for } x \in I_i,\ 1 \leqslant i \leqslant m.$$

It exchanges the intervals I_i according to the permutation τ.

The transformation T is invertible and its inverse is given by the $(\lambda^\tau, \tau^{-1})$ interval exchange transformation.

The symbolic coding of $x \in I$ is $\omega_n(x) = i$ if $T^n(x) \in I_i$. This induces a subshift over the alphabet $\mathcal{A}_m$: $\Omega_{\lambda,\tau} = \overline{\{\omega(x):\ x \in I\}}$.

Sturmian subshifts correspond to the case of two intervals as a first return map construction shows.

Keane [49] proved that if the orbits of the discontinuities μ_i of T are all infinite and pairwise distinct, then T is minimal. In this case, the coding is one-to-one and the subshift is minimal and aperiodic. This holds in particular if τ is irreducible and λ is irrational. Here, τ is called irreducible if $\tau(\{1, \dots, k\}) \neq (\{1, \dots, k\})$ for every $k < m$ and λ is called irrational if the λ_i are rationally independent.

Regarding property (B), Boshernitzan has proved two results. First, in [12] the following is shown:

Theorem 5 *(*Boshernitzan*). For every irreducible $\tau \in S_m$ and for Lebesgue almost every λ, the subshift $\Omega_{\lambda,\tau}$ satisfies* (B).

In fact, Boshernitzan shows that for every irreducible $\tau \in S_m$ and for Lebesgue almost every λ, the subshift $\Omega_{\lambda,\tau}$ satisfies a stronger condition where the sequence of n's for which $\eta(n)$ is large cannot be too sparse. This condition is easily seen to imply (B), and hence the theorem above.

Note that when combined with Keane's minimality result, Theorem 5 implies that almost every subshift arising from an interval exchange transformation is uniquely ergodic. The latter statement confirms a conjecture of Keane [49] and had earlier been proven by different methods by Masur [63] and Veech [79]. Keane had in fact conjectured that all minimal interval exchange transformations would give rise to a uniquely ergodic system. This was disproved by Keynes and Newton [51] using five intervals, and then by Keane [50] using four intervals (the smallest possible number). The conjecture was therefore modified in [50] and then ultimately proven by Masur and Veech.

In a different paper, [13], Boshernitzan singles out an explicit class of subshifts arising from interval exchange transformations that satisfy (B). The transformation T is said to be of (rational) rank k if the λ_j span a k-dimensional space over $\mathbb{Q}$ (the field of rational numbers).

Theorem 6 *(*Boshernitzan*). If T has rank* 2*, the subshift $\Omega_{\lambda,\tau}$ satisfies* (B).

5. Circle maps

Let $\alpha \in (0, 1)$ be irrational and $\beta \in (0, 1)$ arbitrary. The coding of the rotation R_α according to a partition into two half-open intervals of length β and $1 - \beta$, respectively, is given by the sequences:

$$v_n(\alpha, \beta, \theta) = \chi_{[0,\beta)}\big(R_\alpha^n \theta\big).$$

We obtain a subshift:

$$\Omega_{\alpha,\beta} = \overline{\big\{v(\alpha, \beta, \theta):\ \theta \in [0, 1)\big\}} \subset \{0, 1\}^{\mathbb{Z}}. \tag{10}$$

Subshifts generated this way are usually called circle map subshifts or subshifts generated by the coding of a rotation. These natural generalizations of Sturmian subshifts were studied, for example, in [1,2,11,31–33,43,48,75].

To the best of our knowledge, the Boshernitzan condition for this class of subshifts has not been studied explicitly. It is, however, intimately related to classical results on inhomogeneous Diophantine approximation problems. In this section we make this connection explicit and study the condition (B) for circle map subshifts.

To describe the relation of frequencies of finite words occurring in a subshift to the length of intervals on the circle, let us write, in analogy to the Sturmian case, $I_0 = [0, \beta)$ and $I_1 = [\beta, 1)$. The word $w = w_1 \dots w_n \in \{0, 1\}^n$ occurs in $v(\alpha, \beta, \theta)$ at site $k + 1$ if and only if

$$R_\alpha^k(\theta) \in I(w_1, \dots, w_n) := \bigcap_{j=1}^{n} R_\alpha^{-j}(I_{w_j}).$$

Thus the frequency of w is θ-independent and equal to the Lebesgue measure of $I(w_1, \dots, w_n)$. Moreover, $I(w_1, \dots, w_n)$ is an interval whose boundary points are elements of the set:

$$P_n(\alpha, \beta) := \bigl\{\{-j\alpha + k\beta\}\colon\ 1 \leqslant j \leqslant n,\ 0 \leqslant k \leqslant 1\bigr\}.$$

This shows in particular that $\Omega_{\alpha,\beta}$ is quasi-Sturmian when $\beta \in \mathbb{Z} + \alpha\mathbb{Z}$ as in this case $P_n(\alpha, \beta)$ splits the unit interval into $n + k$ subintervals for large n. On the other hand, when $\beta \notin \mathbb{Z} + \alpha\mathbb{Z}$, $P_n(\alpha, \beta)$ contains $2n$ elements and the complexity of $\Omega_{\alpha,\beta}$ is $p(n) = 2n$ for n large enough.

Again, the points of $P_n(\alpha, \beta)$ partition the torus into $2n$ (resp., $n + k$) subintervals and hence the length $h_n(\alpha, \beta)$ of the smallest of these intervals bounds the frequency of a factor of length n from below. Explicitly, we have:

$$h_n(\alpha, \beta) = \min\bigl\{\|q\alpha + r\beta\|\colon\ 0 \leqslant |q| \leqslant n,\ 0 \leqslant r \leqslant 1,\ (q, r) \neq (0, 0)\bigr\}.$$

Let us also define:

$$\tilde{h}_n(\alpha, \beta) = \min\bigl\{\|q\alpha + \beta\|\colon\ 0 \leqslant |q| \leqslant n\bigr\}.$$

Then $h_n(\alpha, \beta) \leqslant \tilde{h}_n(\alpha, \beta)$ and therefore

$$\limsup_{n\to\infty} n\tilde{h}_n(\alpha, \beta) = 0 \quad \Rightarrow \quad \limsup_{n\to\infty} nh_n(\alpha, \beta) = 0. \tag{11}$$

Since we saw in Theorem 4 above that the points of $P_n(\alpha)$ are nicely spaced for many values of n, the Boshernitzan condition can only fail for a circle map subshift $\Omega_{\alpha,\beta}$ if the orbit of the α-rotation comes too close to β. In other words, to prove such a negative result for a circle map subshift, it should be sufficient to study $\tilde{h}_n(\alpha, \beta)$, followed by an application of (11).

Motivated by Hardy and Littlewood [40], Morimoto [64,65] carried out an in-depth analysis of the asymptotic behavior of the numbers $\tilde{h}_n(\alpha, \beta)$. Morimoto's results and related ones were summarized in [54]. While it is possible to deduce consequences regarding the Boshernitzan condition from these papers, we choose to give direct and elementary proofs of our positive results below and make reference to a specific theorem of Morimoto only for a complementary negative result.

Our first result shows that the Boshernitzan condition holds in almost all cases.

Theorem 7. *Let $\alpha \in (0, 1)$ be irrational. Then the subshift $\Omega_{\alpha,\beta}$ satisfies* (B) *for Lebesgue almost every $\beta \in (0, 1)$.*

Proof. Denote the set of β's for which the Boshernitzan condition fails by $N(\alpha)$,

$$N(\alpha) = \bigl\{\beta \in (0, 1)\colon\ \Omega_{\alpha,\beta} \text{ does not satisfy (B)}\bigr\}.$$

By (9) and Theorem 4, there exists a sequence $n_k \to \infty$ such that

$$\liminf_{k\to\infty} n_k h_{n_k}(\alpha) = C > 0.$$

Let $\varepsilon > 0$ with $\varepsilon < C$ be given and denote the $\frac{\varepsilon}{2n}$-neighborhood of the set $\{\{q\alpha\}\colon\ 0 < |q| \leqslant n\}$ by $U(\varepsilon, n)$. Clearly, every $\beta \in N(\alpha)$ belongs to $U(\varepsilon, n_k)$ for $k \geqslant k_0(\beta)$. Therefore,

$$N(\alpha) \subseteq \liminf_{k\to\infty} U(\varepsilon, n_k) = \bigcup_{m=1}^{\infty} \bigcap_{k\geqslant m} U(\varepsilon, n_k). \tag{12}$$

The sets

$$S_m = \bigcap_{k\geqslant m} U(\varepsilon, n_k)$$

obey $S_m \subseteq S_{m+1}$ and $|S_m| \leqslant \varepsilon$ for every m; $|\cdot|$ denoting Lebesgue measure. Hence,

$$\Bigl|\liminf_{k\to\infty} U(\varepsilon, n_k)\Bigr| \leqslant \varepsilon.$$

It follows that $N(\alpha)$ has zero Lebesgue measure. □

The next result concerns a subclass of α's for which the Boshernitzan condition holds for all β's.

Theorem 8. *Let $\alpha \in (0,1)$ be irrational with bounded continued fraction coefficients, that is, $a_n \leqslant C$. Then, $\Omega_{\alpha,\beta}$ satisfies* (B) *for every $\beta \in (0,1)$.*

Proof. By Lemma 3.1 and (7), we have:

$$h_n(\alpha) > \frac{1}{q_k + q_{k+1}},$$

where k is chosen such that $q_k \leqslant n < q_{k+1}$. Thus, for every n, we have:

$$nh_n(\alpha) > \frac{n}{q_k + q_{k+1}} \geqslant \frac{q_k}{(a_{k+1}+2)q_k} \geqslant \frac{1}{C+2}. \tag{13}$$

Now assume there exists $\beta \in (0,1)$ such that $\Omega_{\alpha,\beta}$ does not satisfy (B). Let $\varepsilon = (7C+14)^{-1}$. As $\limsup_{n\to\infty} n\varepsilon(n) = 0$, we have $n\varepsilon(n) < \varepsilon$ for every sufficiently large n. Thus, for each such n we can find a word of length n with frequency less than ε/n. Now, each such word corresponds to an interval with length less than ε/n with boundary points in $P_n(\alpha,\beta)$. Moreover, invoking (13) and the fact that $\varepsilon < 1/(C+2)$, we infer that the length of the interval has the form $|m_n\alpha - \beta - k_n|$ with $|m_n| \leqslant n$. To summarize, we see that for every n large enough there exist k_n, m_n with $|m_n| \leqslant n$ such that

$$|m_n\alpha - \beta - k_n| \leqslant \frac{\varepsilon}{n}.$$

Clearly, the mapping $n \mapsto m_n$ can take on each value only finitely many times. Therefore, there exists a sequence $n_j \to \infty$ such that $m_{n_j} \neq m_{n_j+1}$. This implies:

$$\bigl|(m_{n_j+1} - m_{n_j})\alpha - (k_{n_j+1} - k_{n_j})\bigr| \leqslant |m_{n_j+1}\alpha - \beta - k_{n_j+1}| + |m_{n_j}\alpha - \beta - k_{n_j}| \leqslant \frac{\varepsilon}{n_j+1} + \frac{\varepsilon}{n_j} \leqslant \frac{2\varepsilon}{n_j}.$$

Since $0 < |m_{n_j+1} - m_{n_j}| \leqslant 2(n_j+1) \leqslant 3n_j =: \tilde{n}_j$, we obtain $\tilde{n}_j h_{\tilde{n}_j}(\alpha) \leqslant 6\varepsilon < (C+2)^{-1}$, which contradicts (13). □

This raises the question whether $\Omega_{\alpha,\beta}$ satisfies (B) for every β also in the case where α has unbounded coefficients a_n. It is a consequence of a result of Morimoto [65] that this is not the case.

Theorem 9 *(*Morimoto*). Let $\alpha \in (0,1)$ be irrational with unbounded continued fraction coefficients. Then, there exists $\beta \in (0,1)$ such that*

$$\limsup_{n\to\infty} n\tilde{h}_n(\alpha,\beta) = 0.$$

Corollary 2. *Let $\alpha \in (0,1)$ be irrational with unbounded continued fraction coefficients. Then, there exists $\beta \in (0,1)$ such that $\Omega_{\alpha,\beta}$ does not satisfy* (B).

Proof. This is an immediate consequence of Theorem 9 and (11). □

Remark. Theorem 7, together with Theorem 1, shows zero-measure Cantor spectrum for almost all subshifts $\Omega_{\alpha,\beta}$ in Lebesgue sense. All previous results for operators with potentials generated by a circle map were restricted to a zero-measure set of parameter values, as we will now explain. Proofs of zero-measure spectrum for the associated operators based on trace map dynamics were given in [9,23,76,77]. They cover the case of arbitrary irrational $\alpha \in (0,1)$ and β's in $(0,1)$ of the form $\beta = m\alpha + n$. This is clearly a zero-measure set in $(0,1) \times (0,1)$. On the other hand, the paper [1] applies the results of [59] and shows zero-measure spectrum for a class of circle map subshifts that is characterized by means of a generalized continued fraction algorithm. It is not hard to show that the cases covered by [1] form a set of zero measure, too.

We close this section with a brief discussion of the case where the circle is partitioned into a finite number of half-open intervals. To be specific, let $0 < \beta_1 < \cdots < \beta_{p-1} < 1$ and associate the intervals of the induced partition with p real values: Let $\beta_p = \beta_0 = 0$, and

$$v_n(\theta) = v_k \quad \Leftrightarrow \quad R_\alpha^n(\theta) \in [\beta_k, \beta_{k+1}).$$

We obtain a subshift over the alphabet $\{v_1, \ldots, v_p\}$,

$$\Omega_{\alpha,\beta} = \overline{\{v(\theta)\colon \theta \in [0,1)\}}.$$

Again, the word $w = w_1 \ldots w_n \in \{0,1\}^n$ occurs in $v(\theta)$ at site $k+1$ if and only if

$$R_\alpha^k \theta \in I(w_1, \ldots, w_n) := \bigcap_{j=1}^{n} R_\alpha^{-j}(I_{w_j})$$

and the connected components of the sets $I(w_1, \ldots, w_n)$ are bounded by the points

$$\{-j\alpha + \beta_k\colon 1 \leqslant j \leqslant n,\ 0 \leqslant k \leqslant p-1\}. \tag{14}$$

The first result shows that (B) holds for a dense set of cases:

Theorem 10. *Let $\alpha \in (0,1)$ be irrational and suppose that $\beta_1, \ldots, \beta_{p-1} \in \mathbb{Q}$. Then the subshift $(\Omega_{\alpha,\beta}, T)$ satisfies the Boshernitzan condition* (B).

Proof. By assumption, the subshift $(\Omega_{\alpha,\beta}, T)$ may be described by a rank 2 IET so that Theorem 6 applies and yields the claim.

Explicitly, $1-\alpha$ is irrational and hence does not belong to the set $\{\beta_1, \ldots, \beta_{p-1}\}$. Thus, there is a unique index j such that $\beta_j < 1-\alpha < \beta_{j+1}$. Now let $\mu_k = \beta_k$ for $1 \leqslant k \leqslant j$, $\mu_{j+1} = 1-\alpha$, and $\mu_k = \beta_{k-1}$ for $j+2 \leqslant k \leqslant p$. The permutation τ of the resulting intervals is induced by the exchange of $[0, 1-\alpha)$ and $[1-\alpha, 1)$. The resulting IET subshift has rank 2 and hence satisfies the Boshernitzan condition (B) by Theorem 6. The subshift $(\Omega_{\alpha,\beta}, T)$ arises from the IET subshift constructed above via the substitution

$$\begin{aligned} k &\mapsto v_{k-1}, \quad 1 \leqslant k \leqslant j+1, \\ k &\mapsto v_{k-2}, \quad j+2 \leqslant k \leqslant p+1 \end{aligned}$$

and hence satisfies the Boshernitzan condition (B) as well by Theorem 2. □

Let us now consider the general case, where we do not assume rationality of the points of the partition. Recall that $\limsup_{n\to\infty} n\tilde{h}_n(\alpha,\beta)$ is an important quantity in the case of a partition of the circle into two intervals. In fact, we showed that this quantity being positive is a necessary condition for (B) to hold. When there are three or more intervals, however, we will need to require a much stronger condition as the β_i's may now "take turns" in being well approximated by the α-orbit. Indeed, we shall now be interested in studying $\liminf_{n\to\infty} n\tilde{h}_n(\alpha,\gamma)$ (for certain values of γ, associated with the β_i's). More precisely, define the following quantity:

$$M(\alpha,\gamma) = \liminf_{|n|\to\infty} |n| \cdot \|n\alpha - \gamma\|.$$

Let,

$$P(\alpha) = \{\gamma\colon M(\alpha,\gamma) > 0\}.$$

Then, we have the following result:

Theorem 11. *Let $\alpha \in (0,1)$ be irrational. Suppose that $0 = \beta_0 < \beta_1 < \cdots < \beta_{p-1} < \beta_p = 1$ are such that*

$$\beta_k - \beta_l \in P(\alpha) \quad \textit{for } 0 \leqslant k \neq l \leqslant p-1.$$

Then the subshift $(\Omega_{\alpha,\beta}, T)$ satisfies the Boshernitzan condition (B).

Remarks. 1. This gives a finite number of conditions whose combination is a sufficient condition for (B) to hold.

2. The set $P(\alpha)$ is non-empty for every irrational α. In fact, for every irrational α there exists a suitable γ such that $M(\alpha,\gamma) > 1/32$; compare [74, Theorem IV.9.3].

3. $M(\alpha,\gamma)$ can be computed with the help of the so-called *negative continued fraction expansion* of α and the *α-expansion* of γ; see [70].

Proof. By (14), all frequencies of words of length n are bounded from below by:

$$\hat{h}_n(\alpha,\beta) = \min\bigl\{\|q\alpha + \beta_k - \beta_l\|\colon\ 0 \leqslant |q| \leqslant n,\ 0 \leqslant k,l \leqslant p-1,\ (q,k-l) \neq (0,0)\bigr\}.$$

As in our considerations above, we choose a sequence $n_k \to \infty$ such that

$$\liminf_{k\to\infty} n_k h_{n_k}(\alpha) = C > 0.$$

By assumption, we have:

$$D = \min\bigl\{M(\alpha, \beta_k - \beta_l)\colon\ 0 \leqslant k \neq l \leqslant p-1\bigr\} > 0.$$

Notice that with these choices of C and D, frequencies of words of length n_k are bounded from below by:

$$\hat{h}_{n_k}(\alpha,\beta) \geqslant \min\left\{\frac{C - \mathrm{o}(1)}{n_k}, \frac{D - \mathrm{o}(1)}{n_k}\right\}.$$

Putting everything together, we obtain:

$$\limsup_{n\to\infty} n\cdot\eta(n) \geqslant \liminf_{k\to\infty} n_k \cdot \eta(n_k) \geqslant \min\{C,D\} > 0,$$

and hence (B) is satisfied. □

6. Arnoux–Rauzy subshifts and episturmian subshifts

In this section we consider another natural generalization of Sturmian subshifts, namely, Arnoux–Rauzy subshifts and, more generally, episturmian subshifts. These subshifts were studied, for example, in [4,30,34,46,47,73,81]. They share with Sturmian subshifts the fact that, for each n, there is a unique subword of length n that has multiple extensions to the right. Our main results will show that, similarly to the circle map case, the Boshernitzan condition is almost always satisfied, but not always.

Let us consider a minimal subshift (Ω, T) over the alphabet $\mathcal{A}_m = \{1,2,\ldots,m\}$, where $m \geqslant 2$. Recall that the set of subwords of length n occurring in elements of Ω is denoted by $\mathcal{W}_n(\Omega)$ (cf. (2)) and that the complexity function p is defined by $p(n) = \sharp\mathcal{W}_n(\Omega)$ (cf. (4)). A word $w \in \mathcal{W}(\Omega)$ is called *right-special* (resp., *left-special*) if there are distinct symbols $a,b \in \mathcal{A}_m$ such that $wa, wb \in \mathcal{W}(\Omega)$ (resp., $aw, bw \in \mathcal{W}(\Omega)$). A word that is both right-special and left-special is called *bispecial*.

For later use, let us recall the *Rauzy graphs* that are associated with $\mathcal{W}(\Omega)$. For each n, we consider the directed graph $\mathcal{R}_n = (V_n, A_n)$, where the vertex set is given by $V_n = \mathcal{W}_n(\Omega)$, and A_n contains the arc from aw to wb, $a,b \in \mathcal{A}_m$, $|w| = n-1$, if and only if $awb \in \mathcal{W}_{n+1}(\Omega)$. That is, $|V_n| = p(n)$ and $|A_n| = p(n+1)$. Moreover, a word is right-special (resp., left-special) if and only if its out-degree (resp., in-degree) is $\geqslant 2$.

Note that the complexity function of a Sturmian subshift obeys $p(n+1) - p(n) = 1$ for every n and hence for every length, there is a unique right-special factor and a unique left-special factor, each having exactly two extensions. This property is clearly characteristic for a Sturmian subshift.

Arnoux–Rauzy subshifts and episturmian subshifts relax this restriction on the possible extensions somewhat, and they are defined as follows: Ω is called an *Arnoux–Rauzy subshift* if for every n, there is a unique right-special word

$r_n \in \mathcal{W}(\Omega)$ and a unique left-special word $l_n \in \mathcal{W}(\Omega)$, both having exactly m extensions. This implies in particular that $p(1) = m$ and hence

$$p(n) = (m-1)n + 1.$$

Arnoux–Rauzy subshifts over $\mathcal{A}_2$ are exactly the Sturmian subshifts.

On the other hand, Ω is called *episturmian* if $\mathcal{W}(\Omega)$ is closed under reversal (i.e., for every $w = w_1 \dots w_n \in \mathcal{W}(\Omega)$, we have $w^R = w_n \dots w_1 \in \mathcal{W}(\Omega)$) and for every n, there is exactly one right-special word $r_n \in \mathcal{W}(\Omega)$.

It is easy to see that every Arnoux–Rauzy subshift is episturmian. On the other hand, every episturmian subshift turns out to be a morphic image of some Arnoux–Rauzy subshift. We shall explain this connection below. Since we are interested in studying the Boshernitzan condition, this fact is important and allows us to limit our attention to the Arnoux–Rauzy case.

Risley and Zamboni [73] found two useful descriptions of a given Arnoux–Rauzy subshift, namely, in terms of the recursive structure of the bispecial words and in terms of an S-adic system.

Let ε be the empty word and let $\{\varepsilon = w_1, w_2, \dots\}$ be the set of all bispecial words in $\mathcal{W}(\Omega)$, ordered so that $0 = |w_1| < |w_2| < \cdots$. Let $I = \{i_n\}$ be the sequence of elements i_n of $\mathcal{A}_m$ so that $w_n i_n$ is left-special. The sequence I is called the *index sequence* associated with Ω. Risley and Zamboni prove that, for every n, w_{n+1} is the *palindromic closure* $(w_n i_n)^+$ of $w_n i_n$, that is, the shortest palindrome that has $w_n i_n$ as a prefix. Conversely, given any sequence I, one can associate a subshift Ω as follows: Start with $w_1 = \varepsilon$ and define w_n inductively by $w_{n+1} = (w_n i_n)^+$. The sequence of words $\{w_n\}$ has a unique one-sided infinite limit $w_\infty \in \mathcal{A}_m^{\mathbb{N}}$, called the *characteristic sequence*, which then gives rise to the subshift $(\Omega(I), T)$ in the standard way; $\Omega(I)$ consists of all two-sided infinite sequences whose subwords occur in w. Risley and Zamboni prove the following characterization.

Proposition 6.1 *(*Risley–Zamboni*). For every Arnoux–Rauzy subshift (Ω, T) over $\mathcal{A}_m$, every $a \in \mathcal{A}_m$ occurs in the index sequence $\{i_n\}$ infinitely many times and $\Omega = \Omega(I)$. Conversely, for every sequence $\{i_n\} \in \mathcal{A}_m^{\mathbb{N}}$ such that every $a \in \mathcal{A}_m$ occurs in $\{i_n\}$ infinitely many times, $(\Omega(I), T)$ is an Arnoux–Rauzy subshift and $\{i_n\}$ is its index sequence.*

The S-adic description of an Arnoux–Rauzy subshift, that is, involving iterated morphisms chosen from a finite set, found in [73] reads as follows.

Proposition 6.2 *(*Risley–Zamboni*). Let (Ω, T) be an Arnoux–Rauzy subshift over $\mathcal{A}_m$ and $\{i_n\}$ the associated index sequence. For each $a \in \mathcal{A}_m$, define the morphism τ_a by*:

$$\tau_a(a) = a \quad \textit{and} \quad \tau_a(b) = ab \quad \textit{for } b \in \mathcal{A}_m \setminus \{a\}.$$

Then for every $a \in \mathcal{A}_m$, the characteristic sequence is given by:

$$\lim_{m \to \infty} \tau_{i_1} \circ \cdots \circ \tau_{i_m}(a).$$

We can now state our positive result regarding the Boshernitzan condition for Arnoux–Rauzy subshifts.

Theorem 12. *Let (Ω, T) be an Arnoux–Rauzy subshift over $\mathcal{A}_m$ and $\{i_n\}$ the associated index sequence. Suppose there is $N \in \mathbb{N}$ such that for a sequence $k_j \to \infty$, each of the words $i_{k_j} \dots i_{k_j+N-1}$ contains all symbols from $\mathcal{A}_m$. Then the Boshernitzan condition* (B) *holds.*

This result is similar to Theorem 7 in the sense that if we put any probability measure ν on $\mathcal{A}_m$ assigning positive weight to each symbol, then almost all sequences $\{i_n\}$ with respect to the product measure $\nu^{\mathbb{N}}$ correspond to Arnoux–Rauzy subshifts that satisfy the assumption of Theorem 12.

Before proving this theorem, we state our negative result, which is an analog of Corollary 2.

Theorem 13. *For every $m \geqslant 3$, there exists an Arnoux–Rauzy subshift over $\mathcal{A}_m$ that does not satisfy the Boshernitzan condition* (B)*.*

Remark 1. The assumption $m \geqslant 3$ is of course necessary since the case $m = 1$ is trivial and the case $m = 2$ corresponds to the Sturmian case, where the Boshernitzan condition always holds; compare Theorem 4.

The Arnoux–Rauzy subshifts are uniquely ergodic and we set:

$$d(w) \equiv \mu(V_w), \quad w \in \mathcal{W}(\Omega),$$

where, as usual, the unique invariant probability measure is denoted by μ.

Proof of Theorem 12. This proof employs the description of the subshift in terms of the bispecial words; compare Proposition 6.1.

Observe that there is some k_0 such that $|w_k| \leqslant 2|w_{k-1}|$ for every $k \geqslant k_0$. Essentially, we need that $i_1, \ldots, i_{k_0-1}$ contains all symbols from $\mathcal{A}_m$.

Now consider a value of $k \geqslant k_0$ such that $i_k \ldots i_{k+N-1}$ contains all symbols from $\mathcal{A}_m$. We claim that

$$|w_k| \cdot \eta\big(|w_k|\big) \geqslant 2^{-N}. \tag{15}$$

By the assumption, this implies

$$\limsup_{n\to\infty} n \cdot \eta(n) \geqslant 2^{-N}$$

and hence the Boshernitzan condition (B).

The Rauzy graph $\mathcal{R}_{|w_k|}$ has one vertex (namely, w_k) with in-degree and out-degree m, while all other vertices have in-degree and out-degree 1. Thus, the graph splits up into m loops that all contain w_k and are pairwise disjoint otherwise. These loops can be indexed in an obvious way by the elements of the alphabet $\mathcal{A}_m$.

Since $w_{k+1} = (w_k i_k)^+$, w_{k+1} begins and ends with w_k and, moreover, w_{k+1} contains all words that correspond to the loop in $\mathcal{R}_{|w_k|}$ indexed by i_k. Iterating this argument, we see that w_{k+N} contains the words from all loops and hence all words from $\mathcal{W}_{|w_k|}(\Omega)$. This implies

$$\min_{w\in\mathcal{W}_{|w_k|}(\Omega)} d(w) \geqslant d(w_{k+N}) \geqslant \frac{1}{|w_{k+N}|} \geqslant \frac{1}{2^N|w_k|}$$

and hence (15), finishing the proof. □

Proof of Theorem 13. This proof employs the description of the subshift in terms of an S-adic structure; compare Proposition 6.2.

We shall construct an index sequence $\{i_n\}$ over three symbols (i.e., over the alphabet $\mathcal{A}_3$) such that the corresponding Arnoux–Rauzy subshift does not satisfy the Boshernitzan condition (B). It is easy to verify that the same idea can be used to construct such a subshift over $\mathcal{A}_m$ for any $m \geqslant 3$.

The index sequence will have the form:

$$i_1 i_2 i_3 \ldots = 1^{a_1} 2^{a_2} 3^{a_3} 1^{a_4} 2^{a_5} 3^{a_6} 1^{a_7} \ldots, \tag{16}$$

with a rapidly increasing sequence of integers, $\{a_n\}$.

By the special form of the Rauzy graph, the words $w_k a$ label all the frequencies of words in $\mathcal{W}_{|w_k|+1}(\Omega)$ since words corresponding to arcs on a given loop in $\mathcal{R}_{|w_k|}$ must have the same frequency. Put differently,

$$\eta\big(|w_k|+1\big) = \min_{a\in\mathcal{A}_3} d_{w_\infty}(w_k a). \tag{17}$$

Here, we make the dependence of the frequency on w_∞ explicit.

Moreover, it is sufficient to control $\eta(n)$ for these special values of n since every subword u that is not bispecial has a unique extension to either the left or the right, and this extension must have the same frequency. This shows:

$$\eta\big(|w_k|+1\big) \geqslant \eta(n) \quad \text{for } |w_k|+1 \leqslant n \leqslant |w_{k+1}|. \tag{18}$$

Now write $\mu_{k,m} = \tau_{i_k} \circ \cdots \circ \tau_{i_{k+m-1}}$. Proposition 6.2 says that the characteristic sequence is given by the limit

$$w = \lim_{m\to\infty} \mu_{1,m}(a) \quad \text{for every } a \in \mathcal{A}_3.$$

We also define:

$$w^{(k)} = \lim_{m\to\infty} \mu_{k,m}(a) = (\mu_{1,k-1})^{-1}(w).$$

By [46], $w^{(k)}$ is the derived sequence labeling the return words of w_k in w_∞. In particular, $w^{(k)}$ labels the occurrences of $w_k a$, $a \in \mathcal{A}_3$, in w. Moreover,

$$d_{w_\infty}(w_k a) = \frac{d_{w^{(k)}}(a)}{\sum_{b\in\mathcal{A}_3} d_{w^{(k)}}(b)|\mu_{1,k-1}(b)|} \leqslant \frac{d_{w^{(k)}}(a)}{\min_{b\in\mathcal{A}_3}|\mu_{1,k-1}(b)|}. \tag{19}$$

Combining (17) and (19), we obtain:

$$\big(|w_k|+1\big)\cdot\eta\big(|w_k|+1\big) \leqslant \frac{|w_k|+1}{\min_{b\in\mathcal{A}_3}|\mu_{1,k-1}(b)|}\cdot \min_{a\in\mathcal{A}_3} d_{w^{(k)}}(a). \tag{20}$$

Notice that $(|w_k|+1)(\min_{b\in\mathcal{A}_3}|\mu_{1,k-1}(b)|)^{-1}$ only depends on $i_1,\dots,i_{k-1}$ and $\min_{a\in\mathcal{A}_3} d_{w^{(k)}}(a)$ only depends on $i_k, i_{k+1},\dots$. Thus, if we choose a rapidly increasing sequence $\{a_n\}$ in (16), we can arrange for

$$\lim_{k\to\infty}\big(|w_k|+1\big)\cdot\eta\big(|w_k|+1\big) = 0. \tag{21}$$

This together with (18) implies:

$$\lim_{n\to\infty} n\cdot\eta(n) = 0,$$

proving the theorem.

Let us briefly comment on (21). Choose a monotonically decreasing sequence $e_k \to 0$. Assign any value $\geqslant 1$ to a_1. Then, a_2 should be chosen large enough so that for $1 \leqslant k \leqslant a_1$, (20) yields

$$\big(|w_k|+1\big)\cdot\eta\big(|w_k|+1\big) \leqslant e_k. \tag{22}$$

Here we use that between consecutive 3's in $w^{(k)}$, there must be at least a_2 2's. Next, we choose a_3 so large that (22) holds for $a_1+1 \leqslant k \leqslant a_2$. Here we use that between consecutive 1's in $w^{(k)}$, there must be at least a_3 3's. We can continue in this fashion, thereby generating a sequence $\{a_n\}$ such that (22) holds for all k. This shows in particular that $(|w_k|+1)\cdot\eta(|w_k|+1)$ can go to zero arbitrarily fast. □

One may wonder what sequences are generated by the procedures described before Propositions 6.1 and 6.2 if one starts with an index sequence that does not necessarily satisfy the assumption above, namely, that all symbols occur infinitely often. It was shown by Droubay, Justin and Pirillo [34,46] that one obtains episturmian subshifts and, conversely, every episturmian subshift can be generated in this way.

Proposition 6.3 (Droubay, Justin, Pirillo). *For every episturmian subshift (Ω, T) over $\mathcal{A}_m$, there exists an index sequence $\{i_n\}$ such that $\Omega = \Omega(I)$. Conversely, for every sequence $\{i_n\} \in \mathcal{A}_m^{\mathbb{N}}$, $(\Omega(I), T)$ is an episturmian subshift and $\{i_n\}$ is its index sequence. For every $a \in \mathcal{A}_m$, the characteristic sequence is given by*

$$\lim_{m\to\infty} \tau_{i_1}\circ\cdots\circ\tau_{i_m}(a).$$

We can now quickly deduce results concerning (B) for episturmian subshifts. If (Ω, T) is an episturmian subshift over $\mathcal{A}_m$, denote by $\mathcal{A} \subseteq \mathcal{A}_m$ the set of all symbols that occur in its index sequence infinitely many times. Fix k such that $i_k, i_{k+1}, \dots$ only contains symbols from $\mathcal{A}$. Thus this tail sequence corresponds to an Arnoux–Rauzy subshift over $|\mathcal{A}|$ symbols and the given episturmian subshift is a morphic image (under $\mu_{1,k-1}$) of it. (Note that $|\mathcal{A}| \geqslant 2$ since (Ω, T) is aperiodic.) If the associated Arnoux–Rauzy subshift satisfies (B) (if, e.g., Theorem 12 applies), then (Ω, T) satisfies (B) by Theorem 2. On the other hand, since every Arnoux–Rauzy subshift is episturmian, Theorem 13 shows that not all episturmian subshifts satisfy (B). In this context, it is interesting to note that Justin and Pirillo showed that all episturmian subshifts are uniquely ergodic [46].

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